Problem Set 4 — RC Circuit Transient Analysis

1. A series RC circuit has $R = 10$ kΩ and $C = 47$ μF, connected to a 12V DC source at $t = 0$. Derive the expression for the voltage across the capacitor $V_C(t)$ during charging.

2. For the circuit in Problem 1, calculate:

(a) The time constant τ

(b) The voltage across the capacitor at $t = 2\tau$

(c) The time required for the capacitor to reach 99% of its final voltage

3. Find the total energy stored in the capacitor at steady state and compare it to the total energy delivered by the source.

4. Sketch $V_C(t)$ and $I(t)$ curves for both the charging phase ($0 \leq t \leq 5\tau$) and the discharging phase. Label key values on each axis.

5. A capacitor initially charged to 12V is connected at $t = 0$ to a network of two resistors: $R_1 = 4$ kΩ in series with the parallel combination of $R_2 = 6$ kΩ and $R_3 = 3$ kΩ. Find the current through R_2 immediately after the switch closes, and find the time constant of the discharge.

1) At $t=0$ the switch closes and KVL around the loop gives $V_s = v_R + v_C$, so

$$v_s = RC\frac{dv_C}{dt} + v_C$$

This is a first-order ODE with $v_C(0)=0$; rearrange and separate:

$$\frac{dv_C}{V_s - v_C} = \frac{dt}{RC}$$

Integrating both sides: $-\ln(V_s - v_C) = t/RC + \text{const}$. Using $v_C(0) = 0$ gives $\text{const} = -\ln V_s$, so

$$v_C(t) = V_s\left(1 - e^{-t/RC}\right) = 12\left(1 - e^{-t/0.47}\right)\ \text{V}$$

where t is in seconds.

2a) The time constant is

$$\tau = RC = (10\times10^3)(47\times10^{-6}) = 0.47\,\text{s}$$

2b) At $t = 2\tau$:

$$v_C(2\tau) = 12(1-e^{-2}) = 12(1-0.1353) = 12(0.8647) \approx 10.38\,\text{V}$$

2c) We need $v_C = 0.99\times12$, so $1 - e^{-t/\tau} = 0.99$, which gives $e^{-t/\tau} = 0.01$. Then

$$t = -\tau\ln(0.01) = 0.47\times4.605 \approx 2.16\,\text{s} \approx 4.6\tau$$

3) Energy stored in the capacitor at steady state ($v_C = 12\,\text{V}$):

$$E_C = \frac{1}{2}CV_s^2 = \frac{1}{2}(47\times10^{-6})(12)^2 = 3.384\times10^{-3}\,\text{J} \approx 3.38\,\text{mJ}$$

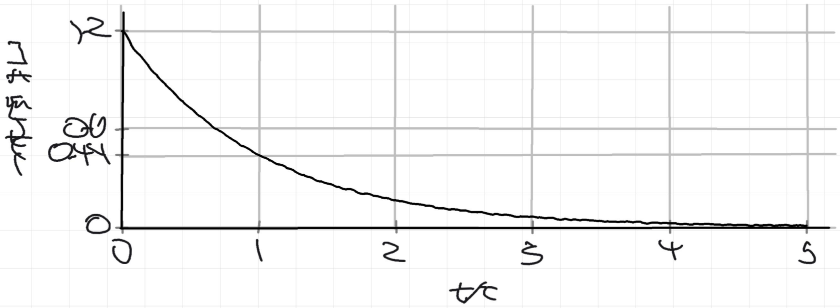
The total charge delivered by the source is $Q = CV_s$, so the total energy from the source is

$$E_s = QV_s = CV_s^2 = (47\times10^{-6})(144) = 6.768\,\text{mJ}$$

So the capacitor stores exactly half the energy delivered by the source. The other half, 3.38 mJ, is dissipated as heat in the resistor. This 50/50 split holds regardless of R.

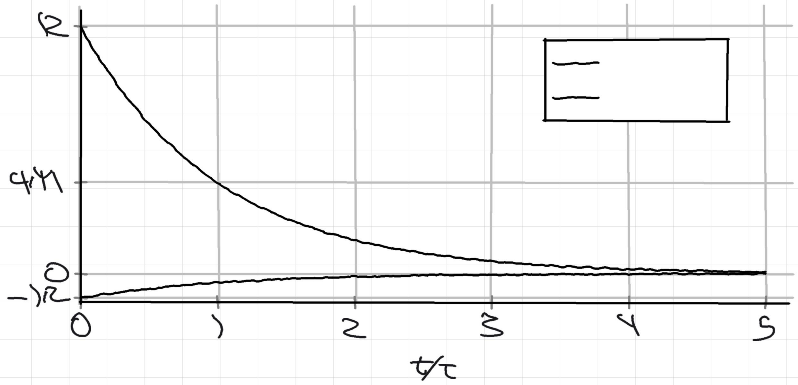
4)

Charging phase ($0 \le t \le 5\tau$): $v_C(t) = 12(1-e^{-t/\tau})$ and $i(t) = \frac{12}{10k}e^{-t/\tau} = 1.2e^{-t/\tau}$ mA



Key values during charging: at $t = 0$, $V_C = 0\ V$ and $I = 1.2\ mA$. At $t = \tau$, $V_C \approx 7.58\ V$ and $I \approx 0.44\ mA$. At $t = 5\tau$, $V_C \approx 11.92\ V$ and $I \approx 0.008\ mA$ (essentially zero).

Discharging phase: capacitor starts at 12 V and decays through R. Now $V_C(t) = 12e^{-t/\tau}$ and $I(t) = -1.2e^{-t/\tau}$ mA (current reverses direction).



At $t = \tau$, $V_C \approx 4.41\ V$. By $t = 5\tau$, $V_C \approx 0.08\ V$ (essentially discharged).

5) First, find the equivalent resistance seen by the capacitor. R_2 and R_3 are in parallel:

$$R_{2\|3} = \frac{R_2 R_3}{R_2 + R_3} = \frac{(6)(3)}{6+3} = 2\ k\Omega$$

This is in series with R_1, so the total resistance is

$$R_{eq} = R_1 + R_{2\|3} = 4 + 2 = 6\ k\Omega$$

At $t = 0^+$, the capacitor still has 12 V across it so the total current leaving the capacitor is

$$I_{total} = \frac{12}{R_{eq}} = \frac{12}{6000} = 2\ mA$$

This 2 mA flows through R_1 and then splits between R_2 and R_3. Using the current divider (current splits inversely with resistance):

$$I_{R_2} = I_{total} \times \frac{R_3}{R_2 + R_3} = 2 \times \frac{3}{9} = \frac{2}{3}\ mA \approx 0.667\ mA$$

The discharge time constant is

$$\tau = R_{eq} C = (6 \times 10^3)(47 \times 10^{-6}) = 0.282\ s$$