Homework 9 — Eigenvalues and Diagonalization

1. Find all eigenvalues of the matrix A:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2. For each eigenvalue found in Problem 1, determine the corresponding eigenvectors.

3. Is the matrix A diagonalizable? If so, find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

4. Using the diagonalization from Problem 3 (or another method if A is not diagonalizable), compute A^5.

5. Let B be a real symmetric 3×3 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 5$. Without computing B directly, find $\det(B)$ and $\mathrm{tr}(B)$.

1) A is upper triangular, so the eigenvalues are just the diagonal entries:

$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 1$

2) For each eigenvalue we solve $(A - \lambda I)x = 0$.

$\lambda = 2$: We row reduce $A - 2I$:

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_2 = 0$, $x_3 = 0$, and x_1 is free. Eigenvector: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\lambda = 3$: Row reduce $A - 3I$:

$$A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = x_2$, $x_3 = 0$, and x_2 is free. Eigenvector: $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$\lambda = 1$: Row reduce $A - I$:

$$A - I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From row 2: $x_2 = -x_3/2$. From row 1: $x_1 = -x_2 = x_3/2$. Setting $x_3 = 2$:

Eigenvector: $v_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

3) We have 3 distinct eigenvalues, so we get 3 linearly independent eigenvectors, which means A is diagonalizable. Setting $P = [v_1 \mid v_2 \mid v_3]$:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $A = PDP^{-1}$.

4) Since $A = PDP^{-1}$, we have $A^5 = PD^5P^{-1}$. First $D^5 = \mathrm{diag}(32, 243, 1)$.

Now we need P^{-1}. Since P is upper triangular, we can find P^{-1} directly:

$$P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

(You can verify $PP^{-1} = I$.) Now we compute PD^5 first:

$$PD^5 = \begin{bmatrix} 32 & 243 & 1 \\ 0 & 243 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Then $A^5 = (PD^5)P^{-1}$:

$$A^5 = \begin{bmatrix} 32 & 243 & 1 \\ 0 & 243 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Computing each entry:

$$A^5 = \begin{bmatrix} 32 & -32 + 243 & -32 + \frac{243}{2} + \frac{1}{2} \\ 0 & 243 & \frac{243}{2} - \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 211 & 90 \\ 0 & 243 & 121 \\ 0 & 0 & 1 \end{bmatrix}$$

5) The determinant is the product of the eigenvalues and the trace is their sum:

$\det(B) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1 \cdot 3 \cdot 5 = 15$

$\mathrm{tr}(B) = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 3 + 5 = 9$

These follow because the characteristic polynomial is $(\lambda - 1)(\lambda - 3)(\lambda - 5)$, and expanding it shows the constant term (up to sign) gives the determinant while the coeffient of λ^2 gives the trace. This works for any square matrix, not just symmetric ones.