Problem Set 7 — Double Integrals in Polar Coordinates

1. Convert the following integral to polar coordinates and evaluate:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

2. Evaluate the double integral over the region R, where R is the semicircular region $x^2 + y^2 \leq 4$, $y \geq 0$:

$$\iint_R \sqrt{x^2 + y^2}\, dA$$

3. Find the area enclosed by one petal of the rose curve $r = \cos(3\theta)$ using a double integral.

4. Use polar coordinates to compute the volume of the solid bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the xy-plane.

5. Evaluate the integral by converting to polar coordinates:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

1) The region of integration is the quarter-disk $x^2 + y^2 \leq 1$ in the first quadrant. In polar coordinates, $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$, with $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx = \int_0^{\pi/2} \int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{\pi/2} \int_0^1 r^3\,dr\,d\theta$$

The inner integral gives $\left(\frac{r^4}{4}\right)\Big|_0^1 = \frac{1}{4}$. Then

$$\int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$$

2) The region R is the upper half of the disk of radius 2, so $0 \leq r \leq 2$, $0 \leq \theta \leq \pi$. Since $\sqrt{x^2 + y^2} = r$,

$$\iint_R \sqrt{x^2 + y^2}\,dA = \int_0^{\pi} \int_0^2 r \cdot r\,dr\,d\theta = \int_0^{\pi} \int_0^2 r^2\,dr\,d\theta$$

The inner integral is $\left(\frac{r^3}{3}\right)\Big|_0^2 = \frac{8}{3}$. So

$$\int_0^{\pi} \frac{8}{3}\,d\theta = \frac{8\pi}{3}$$

3) One petal of $r = \cos(3\theta)$ is traced when $\cos(3\theta) \geq 0$. Taking the petal centered on $\theta = 0$, this runs from $\theta = -\pi/6$ to $\theta = \pi/6$. The area is

$$A = \int_{-\pi/6}^{\pi/6} \int_0^{\cos(3\theta)} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2(3\theta)\,d\theta$$

Using $\cos^2(u) = \frac{1+\cos(2u)}{2}$,

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1 + \cos(6\theta)}{2}\,d\theta = \frac{1}{4}\left[\theta + \frac{\sin(6\theta)}{6}\right]_{-\pi/6}^{\pi/6}$$

Plugging in: $\sin(\pi) = 0$ and $\sin(-\pi) = 0$, so the sine terms vanish. We get

$$A = \frac{1}{4}\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right) = \frac{1}{4} \cdot \frac{\pi}{3} = \frac{\pi}{12}$$

4) The paraboloid $z = 9 - x^2 - y^2$ meets the xy-plane when $9 - r^2 = 0$, so $r = 3$. The volume is

$$V = \int_0^{2\pi} \int_0^3 (9 - r^2)\,r\,dr\,d\theta = 2\pi \int_0^3 (9r - r^3)\,dr$$

$$= 2\pi\left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_0^3 = 2\pi\left(\frac{81}{2} - \frac{81}{4}\right) = 2\pi \cdot \frac{81}{4} = \frac{81\pi}{2}$$

5) Converting to polar: $x^2 + y^2 = r^2$, and the region is all of R^2, so $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

For the inner integral, let $u = r^2$, $du = 2r\,dr$:

$$\int_0^{\infty} r e^{-r^2}\,dr = \frac{1}{2}\int_0^{\infty} e^{-u}\,du = \frac{1}{2}$$

So the whole thing is $2\pi \cdot \frac{1}{2} = \pi$.

This is the classic Gaussian integral result: since the double integral equals $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$, we get $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.